Vector Auto Regressions

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Univariate Processes

- **Auto Regressive Process of Order 1 (AR(1))**: $x_t = \beta x_{t-1} + \epsilon_t \implies$
- $x_t = \epsilon_t + \beta\epsilon_{t-1} + \beta^2\epsilon_{t-2} + \beta^3\epsilon_{t-3} + ... + \beta^j\epsilon_{t-j} + ..$
- **Wold Representation** - moving average representation of AR process $x_t = \sum_{s=0}^{\infty} \beta^s \epsilon_{t-s}$
- $x_t = \sum_{s=0}^{\infty} \beta^s (L)^s \epsilon_t$ (Lagged operator)
- **AR(p) process**: $x_t = c + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \beta_3 x_{t-3} + ...\beta_p x_{t-p} + \epsilon_t$
- Requirements for AR process:
  - **Covariance Stationarity**, i.e.,
    1. $E(x_t) = E(x_0)\forall t$ (mean stationarity, not dependent on t)
    2. $E(x_t x_{t-s})$ is finite and independent of t for all t
  - $|\beta| < 1$, check for **Unit Root** processes

Reduced Form VAR

We start off with estimating a reduced form VAR

- VAR(1):

$$\begin{pmatrix} a_t \\ b_t \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \begin{pmatrix} a_{t-1} \\ b_{t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix}$$

- $X_t = AX_{t-1} + \epsilon_t$
- $X_t = ALX_t + \epsilon_t$
- $(I - AL)X_t = \epsilon_t$
- **Wold Representation:** $X_t = (I - AL)^{-1}\epsilon_t$
- $X_t = C(L)\epsilon_t$, where C(L) is a lagged polynomial
- $X_t = I\epsilon_t + A\epsilon_{t-1} + A^2\epsilon_{t-2} + A^3\epsilon_{t-3}.....$
  (MA Coefficients $= I, A, A^2...$)
- **Impulse Response Functions** : Response of a shock in ϵ_t to X_{t+s}
  1. E.g. Contemporaneous Impact: $\frac{\delta X_t}{\delta \epsilon_t} = I$.
  2. E.g. s periods from now, $\frac{\delta X_{t+s}}{\delta \epsilon_t} = A^s$

The main issue with the reduced form VAR impulse response function is as follows:

- We allow shocks to happen to only one variable at a time, yet we allow contemporaenous correlation across errors. Hence, the assumption of independent shocks to one variable is not reasonable

Thus, we want to create new error terms that are orthogonal. One of the simple ways to do this is the Choleski Factorization.

Choleski Factorization

Let the vcov of ϵ_t be Ω. Current Ω has non-zero diagonal elements, indicahting the error terms are correlated.
Since Ω is a real valued, positive definite, symmetric matrix
$\exists P$ such that $(P'P)^{-1} = \Omega$, where is P is lower triangular.
For invertible lower triangular matrix P, P^{-1} is also lower triangular. We pre-multiply ϵ_t with P such that $P\epsilon_t = \nu_t$, such that Ω_ν has 0 on the off diagonal elements indicating no correlation between the error terms.

$$\Omega_\nu = E(P\epsilon_t P'\epsilon_t') \tag{1}$$

$$\Omega_\nu = P(P'P)^{-1}P' \tag{2}$$

$$\Omega_\nu = (PP^{-1})(PP^{-1})' = I \tag{3}$$

$$X_t = C(L)P^{-1}P\epsilon_t \tag{4}$$

$$X_t = D(L)\nu_t \tag{5}$$

Contemporaneously, $A = I$, therefore, $I \times P^{-1}$ is a lower triangular matrix as P^{-1} is lower triangular.

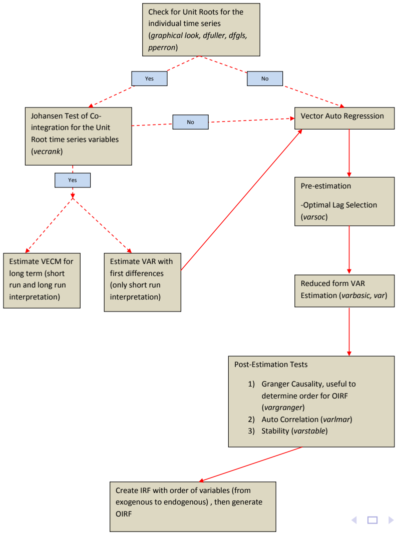
$$\begin{pmatrix} a_t \\ b_t \end{pmatrix} = \begin{pmatrix} d_{11} & 0 \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} \nu_{1t} \\ \nu_{2t} \end{pmatrix}$$

$\implies$ a shock in a_t is only impacted contemporaneously by a shock in itself, while b_t is impacted by shocks in both itself and in a_t contemporaneously.

Note: Order of variables matters - from exogenous to endogenous.

Choleski Factorization is the simplest orthogonalization that can be done. There are several other ones that can be done, including the long run impact restriction by Blanchard and Quah (1989)

Steps to Get Orthogonalized IRF by STATA



Orthogonalized IRFs

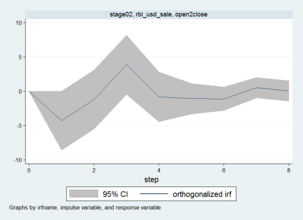


Figure: RBI Int on USDINR(1)

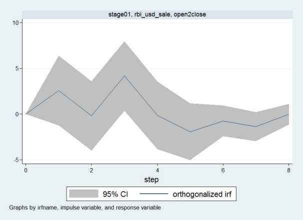


Figure: RBI Int on USDINR(2)