Splitting the Pie

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Question

There are two players; A, and B. They are given by a pie which they must split between themselves. In Period 1, Player A proposes a division, and Player B accepts or rejects this division. If Player B accepts the division proposed by Player A, then the game ends at that time itself and both players walk away with their share of the pie. If Player B rejects, then Player B proposes the divison in Period 2. Now Player A has the option of accepting or rejecting the division. Again the same conditions apply, and the game can continue or end in Period 2. However, for each Period the game goes on for, the size of the pie keeps diminishing by a certain amount.

Introduction

- This is an important game theory question, forms a basis for games based on bargaining
- It also serves as a useful introduction to thinking about *'dynamic'* issues in economics; i.e. issues where you analyze agents' behaviors over multiple time periods

Some Important Concepts

Some important concepts need to be kept in mind for solving this game:

- **Nash Equilibrium (NE)** - a stable state in a game such that no agent has an incentive to deviate to another action
- **Sub Game Perfect Nash Equilibrium (SPNE)** [1] : A NE in every sub-game
- **Backward Induction**: A standard way to solve for SPNE in game theory problems with multiple periods; the concept is to solve for the games by solving from the last period and going back to the first period

[1] A slightly technical point; to understand in greater detail click here

- **Discount Factor (β)** : The factor by which the agents discount future values of the pie. It can be calculated as follows - $\beta = \frac{1}{1+r}$, where r is the rate of interest, which can be taken as an opportunity cost. For example, if r is 100%, β is 0.5, so a value of 100 at the current time period is only valued at 50 in the next time period.

Setting up the Problem

Before, solving the problem, let's set up the framework with some details

- Let's assume that the game lasts for n periods, in the $n+1$ period the value of the pie will be 0; we will test out the game for different levels of n
- Let's normalize the value of the pie to 1
- Let's assume that the pie diminishes in value by $\frac{1}{n}$ for every period it goes on; for example, if the game lasts for 3 periods and it goes on for two periods then the value of the pie left in the second period will be $\frac{2}{3}$; this will be a simplification for the discount factor [2] that was mentioned in the previous slide

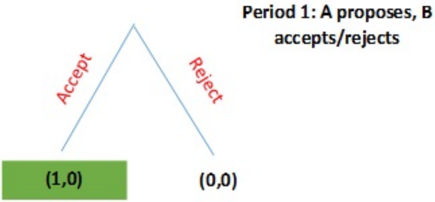
[2] The same logic would apply if we solved with discount factors instead of the pie actually diminishing

$n = 1$

- This is the simplest case, Player A proposes a division and Player B accepts or rejects the division
- If Player B accepts the division then both players walk away with their respective shares, while if Player B rejects then both players get zero and the game ends
- Logically speaking, Player A would want to maximize his share while ensuring that Player B still accepts the division
- Let's say Player A offers a division of $1 - \epsilon$ for himself, while ϵ for Player B; where ϵ is a very very small share of the pie
- Clearly, Player B would accept because by having ϵ he is better off accepting, rather than rejecting

- But for every value of ϵ, there is a small of value like $\frac{\epsilon}{100} = \eta$; and there will be similar smaller values for η and so on
- Therefore, in the limit, the value of ϵ would tend to 0[3]
- Thus, for $n = 1$ the SPNE would be that Player A offers a division of $(1, 0)$ and Player B will accept this divison

[3] It may seem a little counter-intuitive that Player A offers Player B only 0. However, this is a mathematical property that occurs because the value of the pie is in continuous numbers. If the value of the pie was in discrete numbers, then the SPNE would be such that Player A would offer Player B the smallest possible discrete number. For example, if the pie could be divided into divisions of 0.1, then the SPNE would be $(0.9, 0.1)$

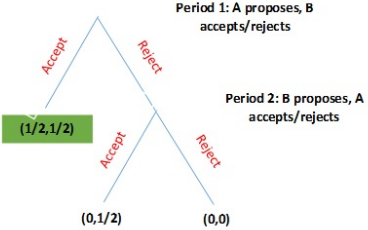

Period 1: A proposes, B accepts/rejects
Accept
Reject
(1,0)
(0,0)

$n = 2$

- We need to solve this game via backward induction
- Let's go to Period 2 first, assuming Player B has rejected Player A's proposed division in Period 1
- The value of the pie left is $\frac{1}{2}$; Player B proposes a division and Player A must accept or reject
- We can use the same logic for the game $n = 1$ to find Player B's proposed division in Period 2
- Player B would like to maximise his share and would propose a division of $(0, \frac{1}{2})$[4], and Player B would accept this division
- So, going back to Period 1, Player A now knows that Player B has to be offered a minimum of $\frac{1}{2}$ to not reject Player A's offer in Period 1

[4] We use the following order for all the proposed divisions (Player A's share, Player B's share)

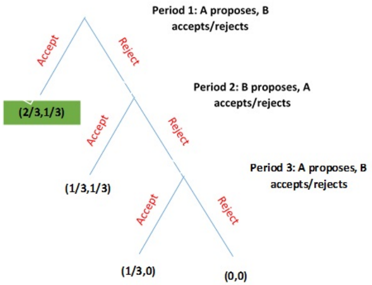
- Therefore, in Period 1, Player A would offer Player B $\frac{1}{2}$ and would keep the rest of the pie $(1 - \frac{1}{2} = \frac{1}{2})$
- Now, Player B would have no reason of rejecting Player A's offer in Period 1
- Thus, for $n = 2$ the SPNE would be that Player A offers a division of $(\frac{1}{2}, \frac{1}{2})$ and Player B will accept this divison in Period 1


Period 1: A proposes, B accepts/rejects
Accept
Reject
Period 2: B proposes, A accepts/rejects
(1/2,1/2)
Accept
Reject
(0,1/2)
(0,0)

$n = 3$

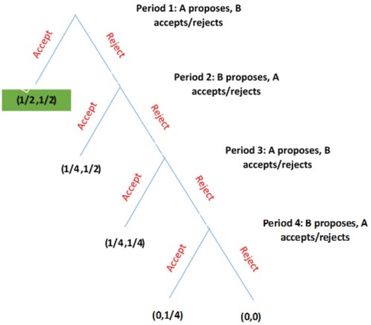
- We need to solve this game via backward induction
- Let's go to Period 3 first
- The value of the pie left is $\frac{1}{3}$; Player A proposes a division and Player B must accept or reject
- We can use the same logic for the game $n = 1$ to find Player A's proposed division in Period 3
- Player A would like to maximise his share and would propose a division of $(\frac{1}{3}, 0)$, and Player B would accept this division
- So, going back to Period 2, Player B now knows that Player A has to be offered a minimum of $\frac{1}{3}$ to not reject Player B's offer in Period 2; Player B would keep the rest of the pie $(\frac{2}{3} - \frac{1}{3} = \frac{1}{3})$; with this division Player A would have not reject Player B's division in Period 2

- Therefore, in Period 1, Player A would offer Player B $\frac{1}{3}$ and would keep the rest of the pie $(1 - \frac{1}{3} = \frac{2}{3})$
- Now, Player B would have no reason of rejecting Player A's offer in Period 1
- Thus, for $n = 3$ the SPNE would be that Player A offers a division of $(\frac{2}{3}, \frac{1}{3})$ and Player B will accept this divison in Period 1


Period 1: A proposes, B accepts/rejects
Accept
Reject
(2/3,1/3)
Period 2: B proposes, A accepts/rejects
Accept
Reject
(1/3,1/3)
Period 3: A proposes, B accepts/rejects
Accept
Reject
(1/3,0)
(0,0)

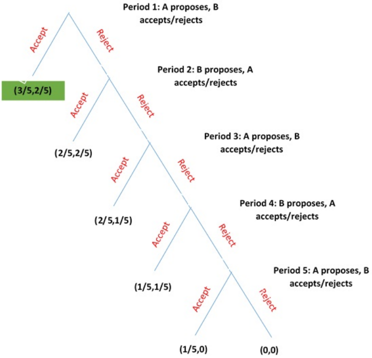
$n = 4$

- For $n = 4$ the SPNE would be that Player A offers a division of $(\frac{1}{2}, \frac{1}{2})$ and Player B will accept this divison in Period 1



$n = 5$

- For $n = 5$ the SPNE would be that Player A offers a division of $(\frac{3}{5}, \frac{2}{5})$ and Player B will accept this divison in Period 1

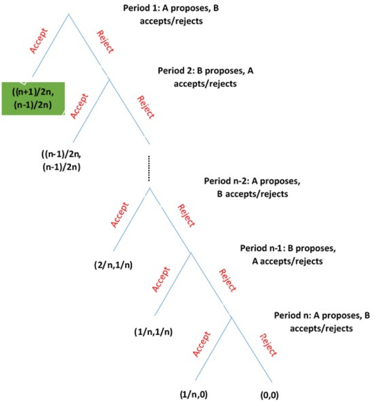


Generalizing Results

- We can clearly see that there is a pattern, and the pattern is distinct for odd and even number of periods

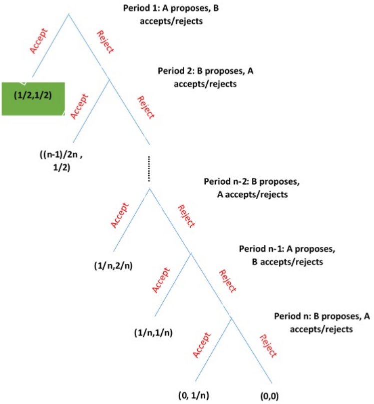
n is Odd

- For n being Odd, the SPNE would be that Player A offers a division of $(\frac{n+1}{2n}, \frac{n-1}{2n})$ and Player B will accept this divison in Period 1



n is Even

- For n being Even, the SPNE would be that Player A offers a division of $(\frac{1}{2}, \frac{1}{2})$ and Player B will accept this divison in Period 1 [5]



[5] As $n \to \infty$, both odd and even cases tend to the same SPNE

Concluding Remarks

- The agreement is always reached in Period 1, irrespective of the length of the game
- There is no wastage, i.e. the value of the pie shared between the two players is always 1
- There are distinct results, depending on whether Player A or Player B is proposing the division in Period n (odd vs even)
- Similar results would hold if we would have considered discount factors